Completeness of IPC with respect to evidence semantics

1. Intuitionistic validity

To say that a logical formula $F(\bar{P})$, such as $P \wedge Q \Rightarrow P$, is <u>valid</u> with respect to the intended evidence semantics for IPC is to say "for all propositions P, Q we know $P \wedge Q \Rightarrow P$." Generally we use this definition.

Definition A formula $F(\bar{P})$ in propositional variables $\bar{P} = P_1, \ldots, P_n$ is <u>(intuitionistically) valid</u> iff $\forall \bar{P}: Prop.\ F(\bar{P})$.
We also say that $F(\bar{P})$ is valid for <u>evidence semantics</u>.

To say that Refinement Logic is complete would be to prove constructively that $\underline{\forall \bar{P}: Prop.\ F(\bar{P}) \Rightarrow \exists\, pf: RefinementProof.\ pf \vdash F(\bar{P})}$.

In 1947 Beth conjectured that this would be true for his tableau notion of an intuitionistic proof. We have not examined his proof system nor this conjecture because in 1962 G. Kreisel, using some results of Gödel, showed that there could not be such a constructive proof using intuitionistic principles known in 1962. This result is too specialized for this course. Moreover, new concepts have arisen in computer science that allowed this problem to be solved in 2011 [Constable & Bickford 11]. We will present these results.

We can appreciate why intuitionistic completeness is hard by observing that if we knew this result and knew that $P \vee \sim P$ is not provable, then we would know $\sim \forall P{:}\,Prop.\,(P \vee \sim P)$. This would give us a semantic reason to deny the Law of Excluded Middle. We will see next that we do know that $P \vee \sim P$ is NOT provable in Refinement Logic. It is also not provable in the Hilbert style logic for IPC presented by Kleene in his book Introduction to Metamathematics. Indeed it is not provable in any known intuitionistic logic.

We will show next that the provable formulas of IPC have a special property, they are uniformly valid. This means that they have evidence terms that provide evidence without knowledge of the exact atomic propositions substituted for the atomic propositional variables, the $P_1, P_2, \ldots$. For example, the evidence for $P \Rightarrow P$ is simply $\lambda x.x$. This evidence is correct independent of which propositions P we have in mind, e.g. P can be $0=0$, $0=1$, Goldbach's conjecture, the Reimann hypothesis, etc. We say that $\lambda x.x$ is "polymorphic"; it works for "many forms." We see this property for all provable formulas, e.g. $\lambda x.x_1$ proves $(P \wedge Q) \Rightarrow P$, and the term does not mention either P or Q explicitly. So the evidence is polymorphic. We say that $(P \wedge Q) \Rightarrow P$ is uniformly valid. We define this concept precise next.

CS 4860 Oct 2, 2012

Lecture 12

2. Uniform validity

A formula $F(\bar{P})$ is uniformly valid iff there is an evidence term (also called a realizer), evd such that $evd \in [F(\bar{P})]$ for all values of $\bar{P}$, i.e. for any atomic propositions. We write this symbolically by using a new kind of quantifier, a uniform quantifier.

We say $\forall[P: Prop]. F(\bar{P})$ precisely when there is a single evidence object evd in each of $[F(\bar{P})]$. We will see later in the course that we can define this quantifier using the notion of an intersection type. Given two types A, B, we define the binary intersection, $A \cap B$, as the objects that belong to both type A and type B. So for two propositions, say

$P \Rightarrow P$ and $Q \Rightarrow Q$, we can consider $[P \Rightarrow P] \cap [Q \Rightarrow Q]$ and we see that $\lambda x.x$ belongs to this type.

We can define $[F(P_1)] \cap [F(P_2)]$ but also there is an infinitary union $\bigcap_{P_i \in Prop} [F(P_i)]$. Notice that $\lambda x.x$ belongs to $\bigcap_{P_i \in Prop} [P_i \Rightarrow P_i]$.

We define $\forall[\bar{P}: Prop]. F(\bar{P})$ as $\bigcap_{\bar{P} \in Prop} [F(\bar{P})]$.

3. Completeness Theorem.

We can now state one version of completeness.

Any uniformly valid formula $F(\bar{P})$ of IPC is provable.

We will prove this version later in the course. At this point we can easily prove the slightly weaker version in which the evidence term uses only logical operators.

Any formula $F(\bar{P})$ which is valid with respect to purely logical evidence is provable.

4. Examples of the proof method

We can see the general idea of the proof method on simple examples such as $P \Rightarrow (P \vee Q)$, $P \Rightarrow (Q \Rightarrow P)$, $P \wedge (Q \vee R) \Rightarrow (P \wedge Q) \vee (P \wedge R)$. The idea is that we can convert the evidence term systematically into a proof.

$$\models P \Rightarrow (Q \Rightarrow P), \ \lambda x.\lambda y.x$$

$$\models P \Rightarrow (Q \Rightarrow P) \ \ ImpR\,x, \ \lambda y.x$$

$$x{:}P \ \models (Q \Rightarrow P), \ \lambda y.x$$

$$x{:}P \ \models (Q \Rightarrow P) \ \ ImpR\,y, \ x$$

$$x{:}P, \ y{:}Q \models P, \ x$$

$$x{:}P, \ y{:}Q \vdash P \ \ hyp\,x$$